JOURNAL OF APPROXIMATION THEORY 36, 183–196 (1982)

Error Analysis for Interpolating Complex Cubic Splines with Deficiency 2

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Communicated by R. C. Buck

Received April 20, 1981

It is very important in applications to approximate complex functions on curves by polynomial splines. For closed contours, especially for the unit circle, there are several references in the literature [2–5]. In practice, we are interested in seeking kinds of splines to approximate a given function, satisfying the following requirements: (1) they ought to be easily constructed; (2) the error between them and the given function as well as those between their derivatives must easily allow accurate estimates; (3) these errors should tend to zero when the norm of the mesh tends to zero. For rectifiable Jordan curves, the class of cubic splines will serve, although the requirements (1) and (2) may cause problems.

Cubic splines with deficiency 2 in the real domain were considered in [1], where it was shown that they satisfy the above requirements, though the error-bounds were not worked out in detail.

In this work, we suggest using interpolating complex cubic splines with deficiency 2 to approximate continuous functions given on a smooth curve L. They have these advantages over those with deficiency 1: it is much easier to construct them and much easier to estimate the error-bounds; there is no superfluous restriction on the mesh when the properties of convergence are studied; and the curve concerned may or may not be closed. In the meantime, the rates of convergence are of the same orders as those with deficiency 1. In addition, even if the curve L considered is arc-wise smooth or the given function is arc-wise continuous, the method used here remains in effect if we consider them separately. Certainly, we gain these advantages by sacrificing the continuity of the derivative of the second order at the knots. However, the jumps at the knots may be made arbitrarily small.

We shall restrict ourselves to $f(t) \in C^1$ at first and consider the case $f(t) \in C$ later.

0021-9045/82/110183-14$02.00/0

1. The Basic Estimations

Let $L = \widehat{t_0 t_1}$ be an open smooth arc in the complex plane and $f(t) \in C^1$ be a given function defined on it. We shall denote the modulus of continuity of $f^{(p)}(t)$ (if it is continuous) with respect to the arc-length by $\omega_p(\delta)$.

The cubic polynomial $S(t)$ with boundary conditions

$$S(t_0) = f(t_0) = y_0, \qquad S(t_1) = f(t_1) = y_1,$$
$$S'(t_0) = f'(t_0) = y_0', \qquad S'(t_1) = f'(t_1) = y_1'$$

is uniquely defined and given by

$$S(t) = y_0 + \frac{\Delta y}{\Delta t}(t - t_0) + \left(y_0' - \frac{\Delta y}{\Delta t}\right)\frac{(t-t_0)(t-t_1)^2}{\Delta t^2} + \left(y_1' - \frac{\Delta y}{\Delta t}\right)\frac{(t-t_0)^2(t-t_1)}{\Delta t^2}, \tag{1.1}$$

where $\Delta y = y_1 - y_0$, $\Delta t = t_1 - t_0$.

We introduce the arc-length parameter $s = s(t)$ on L with $s(t_0) = s_0$, $s(t_1) = s_1$ and denote $\Delta s = s_1 - s_0\,(>0)$. It is well-known [6] that there exists a constant $C \geqslant 1$ such that for every pair of $t, t' \in L$

$$|\widehat{tt'}| \leqslant C\,|t - t'|.$$

We seek to estimate $|f^{(p)}(t) - S^{(p)}(t)|$, $p = 0, 1$. Note that

$$\left|f'(t) - \frac{\Delta y}{\Delta t}\right| = \left|\frac{1}{\Delta t}\int_{t_0}^{t_1}[f'(t) - f'(\tau)]\,d\tau\right|,$$

where the path of integration is taken along L. Thus we have

$$\left|f'(t) - \frac{\Delta y}{\Delta t}\right| \leqslant C\omega_1(\Delta s). \tag{1.2}$$

Let $t^* = t(s^*)$ be the mid-point of L. If $t \in \widehat{t_0 t^*}$, then, by (1.2),

$$\left|f(t) - y_0 - \frac{\Delta y}{\Delta t}(t - t_0)\right| = \left|\int_{t_0}^{t}\left[f'(\tau) - \frac{\Delta y}{\Delta t}\right]d\tau\right| \leqslant \frac{1}{2}C\omega_1(\Delta s)\,\Delta s.$$

Since

$$|(t-t_0)(t-t_1)^2| + |(t-t_0)^2(t-t_1)| \leqslant (s - s_0)(s_1 - s)\,\Delta s \leqslant \tfrac{1}{4}\Delta s^3,$$

we have

$$|f(t)-S(t)| \leqslant (\tfrac{1}{2}C+\tfrac{1}{4}C^3)\,\omega_1(\Delta s)\,\Delta s. \tag{1.3}$$

By symmetry, (1.3) is also valid for $t \in \widehat{t^*t_1}$ and hence for every $t \in L$.

Since

$$S'(t)=\frac{\Delta y}{\Delta t}+\left(y_0'-\frac{\Delta y}{\Delta t}\right)\frac{(t-t_1)(3t-2t_0-t_1)}{\Delta t^2}$$
$$+\left(y_1'-\frac{\Delta y}{\Delta t}\right)\frac{(t-t_0)(3t-t_0-2t_1)}{\Delta t^2},$$

using (1.2) again, we obtain

$$|f'(t)-S'(t)| \leqslant (C+\tfrac{3}{2}C^3)\,\omega_1(\Delta s),$$

in which we have used the inequality

$$|(t-t_1)(3t-2t_0-t_1)|+|(t-t_0)(3t-t_0-2t_1)|$$
$$\leqslant (s_1-s)(s+s_1-2s_0)+(s-s_0)(2s_1-s_0-s) \leqslant \tfrac{3}{2}\Delta s^2.$$

If $f(t) \in C^2$, we may estimate $|f^{(p)}(t)-S^{(p)}(t)|$, $p=0, 1, 2$. Since

$$S''(t)=2\left(y_0'-\frac{\Delta y}{\Delta t}\right)\frac{3t-t_0-2t_1}{\Delta t^2}+2\left(y_1'-\frac{\Delta y}{\Delta t}\right)\frac{3t-2t_0-t_1}{\Delta t^2},$$

$$f''(t)-S''(t)=\frac{2}{\Delta t^3}\left[\Delta y-y_0'\Delta t-\frac{1}{2}f''(t)\,\Delta t^2\right](3t-t_0-2t_1)$$
$$+\frac{2}{\Delta t^3}\left[\Delta y-y_1'\Delta t+\frac{1}{2}f''(t)\,\Delta t^2\right](3t-2t_0-t_1). \tag{1.4}$$

Noting that

$$\begin{aligned}|\Delta y-y_0'\,\Delta t-\tfrac{1}{2}f''(t)\,\Delta t^2| &= \left|\int_{t_0}^{t_1}[f''(\tau)-f''(t)](t_1-\tau)\,d\tau\right| \\ &\leqslant \omega_2(\Delta s)\int_{s_0}^{s_1}(s_1-s)\,ds=\tfrac{1}{2}\omega_2(\Delta s)\,\Delta s^2\end{aligned} \tag{1.5}$$

and a similar estimation for $|\Delta y-y_1'\Delta t+\frac{1}{2}f''(t)\,\Delta t^2|$, we have

$$|f''(t)-S''(t)| \leqslant C^2\omega_2(\Delta s)\frac{1}{\Delta t}\,[|3t-t_0-2t_1|+|3t-2t_0-t_1|]$$
$$\leqslant 3C^3\omega_2(\Delta s).$$

Now we write

$$f(t)-S(t)=\left[f(t)-y_0-y_0'(t-t_0)-\frac{1}{2}y_0''(t-t_0)^2\right]$$
$$+\left[\Delta y-y_0'\Delta t-\frac{1}{2}y_0''\Delta t^2\right]\frac{(t-t_0)^2(t+t_0-2t_1)}{\Delta t^3}$$
$$+\left[\Delta y-y_1'\Delta t+\frac{1}{2}y_1''\Delta t^2\right]\frac{(t-t_0)^2(t-t_1)}{\Delta t^3}$$
$$-\frac{y_1''-y_0''}{2\Delta t}(t-t_0)^2(t-t_1)=I_1+I_2+I_3+I_4, \qquad (1.6)$$

where $y_0''=f''(t_0)$, $y_1''=f''(t_1)$. For $t\in\widehat{t_0t^*}$, analogous to (1.5), we have

$$|I_1|\leqslant\frac{1}{8}\omega_2\left(\frac{\Delta s}{2}\right)\Delta s^2\leqslant\frac{1}{8}\omega_2(\Delta s)\Delta s^2,$$
$$|I_2|+|I_3|\leqslant\frac{1}{2}C^2\omega_2(\Delta s)\frac{1}{|\Delta t|}(s-s_0)^2(3s_1-s_0-2s)$$
$$\leqslant\frac{1}{4}C^3\omega_2(\Delta s)\Delta s^2$$

since

$$\max_{s_0\leqslant s\leqslant s^*}(s-s_0)^2(3s_1-s_0-2s)=\tfrac{1}{2}\Delta s^3; \qquad (1.7)$$
$$|I_4|\leqslant\frac{1}{2}\omega_2(\Delta s)\frac{1}{|\Delta t|}(s-s_0)^2(s_1-s)\leqslant\frac{1}{16}C\omega_2(\Delta s)\Delta s^2$$

since

$$\max_{s_0\leqslant s\leqslant s^*}(s-s_0)^2(s_1-s)=\tfrac{1}{8}\Delta s^3. \qquad (1.8)$$

Hence we obtain

$$|f(t)-S(t)|\leqslant\frac{1}{8}\omega_2\left(\frac{\Delta s}{2}\right)\Delta s^2+\left(\frac{1}{16}C+\frac{1}{4}C^3\right)\omega_2(\Delta s)\Delta s^2$$
$$\leqslant\left(\frac{1}{8}+\frac{1}{16}C+\frac{1}{4}C^3\right)\omega_2(\Delta s)\Delta s^2,$$

which is also valid for every $t\in L$ by symmetry.

Now we write

$$f'(t)-S'(t)=[f'(t)-y_0'-y_0''(t-t_0)] \tag{1.9}$$
$$+\left(\Delta y-y_0'\Delta t-\frac{1}{2}y_0''\Delta t^2\right)\frac{(t-t_0)(3t+t_0-4t_1)}{\Delta t^3}$$
$$+\left(\Delta y-y_1'\Delta t+\frac{1}{2}y_1''\Delta t^2\right)\frac{(t-t_0)(3t-t_0-2t_1)}{\Delta t^3}$$
$$-\frac{y_1''-y_0''}{2\Delta t}(t-t_0)(3t-t_0-2t_1)=J_1+J_2+J_3+J_4.$$

Suppose $t\in \widehat{t_0 t}^*$; then

$$|J_1|\leqslant\frac{1}{2}\omega_2\left(\frac{\Delta s}{2}\right)\Delta s;$$

$$|J_2|+|J_3|\leqslant\frac{1}{2}C^2\omega_2(\Delta t)\frac{1}{|\Delta t|}(s-s_0)\cdot 2(3s_1-s_0-2s)$$
$$\leqslant C^3\omega_2(\Delta s)\Delta s,$$

since

$$\max_{s_0\leqslant s\leqslant s^*}(s-s_0)(3s_1-s_0-2s)=\Delta s^2; \tag{1.10}$$

$$|J_4|\leqslant\frac{1}{2}\omega_2(\Delta s)\frac{1}{|\Delta t|}(s-s_0)(2s_1-s_0-s)\leqslant\frac{3}{8}C\omega_2(\Delta s)\Delta s,$$

since

$$\max_{s_0\leqslant s\leqslant s^*}(s-s_0)(2s_1-s_0-s)=\tfrac{3}{4}\Delta s^2. \tag{1.11}$$

Hence, we have for every $t\in L$

$$|f'(t)-S'(t)|\leqslant\frac{1}{2}\omega_2\left(\frac{\Delta s}{2}\right)\Delta s+\left(\frac{3}{8}C+C^3\right)\omega_2(\Delta s)\Delta s$$
$$\leqslant\left(\frac{1}{2}+\frac{3}{8}C+C^3\right)\omega_2(\Delta s)\Delta s.$$

Let $f(t)\in C^3$. Since

$$S'''(t)=\frac{6}{\Delta t^2}\left(y_0'+y_1'-2\frac{\Delta y}{\Delta t}\right),$$

we may write

$$f'''(t)-S'''(t)=\frac{6}{\Delta t^3}\left[\Delta y-y_0'\Delta t-\frac{y_0''}{2}\Delta t^2-\frac{1}{6}f'''(t)\Delta t^3\right]$$
$$+\frac{6}{\Delta t^3}\left[\Delta y-y_1'\Delta t+\frac{y_1''}{2}\Delta t^2-\frac{1}{6}f'''(t)\Delta t^3\right]$$
$$-\frac{3}{\Delta t}\left[y_1''-y_0''-f'''(t)\Delta t\right]=H_1+H_2+H_3.$$

But

$$|H_1|=\frac{6}{|\Delta t|^3}\left|\int_{t_0}^{t_1}[f'''(\tau)-f'''(t)]\frac{(t_1-\tau)^2}{2}d\tau\right|\leqslant C^3\omega_3(\Delta s), \quad (1.12)$$

$$|H_2|\leqslant C^3\omega_3(\Delta s),$$

and

$$|H_3|\leqslant 3C\omega_3(\Delta s),$$

so that

$$|f'''(t)-S'''(t)|\leqslant(3C+2C^3)\,\omega_3(\Delta s).$$

We then expect better estimates for $|f^{(p)}(t)-S^{(p)}(t)|$, $p=0, 1, 2$. Let us rewrite (1.6) as

$$f(t)-S(t)=\left[f(t)-y_0-y_0'(t-t_0)-\frac{y_0''}{2}(t-t_0)^2-\frac{y_0'''}{6}(t-t_0)^3\right]$$
$$+\left(\Delta y-y_0'\Delta t-\frac{y_0''}{2}\Delta t^2-\frac{y_0'''}{6}\Delta t^3\right)\frac{(t-t_0)^2\,(t+t_0-2t_1)}{\Delta t^3}$$
$$+\left(\Delta y-y_1'\Delta t+\frac{y_1''}{2}\Delta t^2-\frac{y_1'''}{6}\Delta t^3\right)\frac{(t-t_0)^2\,(t-t_1)}{\Delta t^3}$$
$$-(y_1''-y_0''-y_0'''\Delta t)\frac{(t-t_0)^2\,(t-t_1)}{2\Delta t}$$
$$+\frac{y_1'''-y_0'''}{6}(t-t_0)^2\,(t-t_1)=K_1+K_2+K_3+K_4+K_5,$$

where $y_0'''=f'''(t_0)$, $y_1'''=f'''(t_1)$. Suppose $t\in\widehat{t_0t}^*$; then, analogous to (1.12),

$$|K_1|\leqslant\left|\int_{t_0}^{t}[f'''(\tau)-y_0''']\frac{(t-\tau)^2}{2}d\tau\right|\leqslant\frac{1}{48}\,\omega_2\left(\frac{\Delta s}{2}\right)\Delta s^3;$$

by (1.7),

$$|K_2| + |K_3| \leqslant \tfrac{1}{6} C^3 \omega_3(\Delta s)(s - s_0)^2 \, (3s_1 - s_0 - 2s)$$
$$\leqslant \tfrac{1}{12} C^3 \omega_3(\Delta s)\, \Delta s^3;$$

by (1.8),

$$|K_4| \leqslant \tfrac{1}{2} C \omega_3(\Delta s)(s - s_0)^2 \, (s_1 - s) \leqslant \tfrac{1}{16} C \omega_3(\Delta s)\, \Delta s^3;$$
$$|K_5| \leqslant \tfrac{1}{48} \omega_3(\Delta s)\, \Delta s^3.$$

Therefore, for $t \in L$, we have

$$|f(t) - S(t)| \leqslant \frac{1}{48} \omega_2 \left(\frac{\Delta s}{2}\right) \Delta s^3 + \left(\frac{1}{48} + \frac{1}{16} C + \frac{1}{12} C^3\right) \omega_3(\Delta s)\, \Delta s^3$$
$$\leqslant \left(\frac{1}{24} + \frac{1}{16} C + \frac{1}{12} C^3\right) \omega_3(\Delta s)\, \Delta s^3.$$

Now we rewrite (1.9) as

$$\begin{aligned} f'(t) - S'(t) &= \left[f'(t) - y_0' - y_0''(t - t_0) - \frac{y_0'''}{2} (t - t_0)^2 \right] \\ &+ \left(\Delta y - y_0' \Delta t - \frac{y_0''}{2} \Delta t^2 - \frac{y_0'''}{6} \Delta t^3 \right) \frac{(t - t_0)(3t + t_0 - 4t_1)}{\Delta t^3} \\ &+ \left(\Delta y - y_1' \Delta t + \frac{y_1''}{2} \Delta t^2 - \frac{y_1'''}{6} \Delta t^3 \right) \frac{(t - t_0)(3t - t_0 - 2t_1)}{\Delta t^3} \\ &- (y_1'' - y_0'' - y_0''' \Delta t) \frac{(t - t_0)(3t - t_0 - 2t_1)}{2 \Delta t} \\ &+ \frac{y_1''' - y_0'''}{6} (t - t_0)(3t - t_0 - 2t_1) = L_1 + L_2 + L_3 + L_4 + L_5. \end{aligned}$$

Again suppose $t \in \widehat{t_0 t}^*$; then

$$|L_1| = \left| \int_{t_0}^{t} [f'''(\tau) - y_0'''](t - \tau)\, d\tau \right| \leqslant \frac{1}{8} \omega_3 \left(\frac{\Delta s}{2}\right) \Delta s^2;$$

analogous to (1.12), by (1.10),

$$|L_2| + |L_3| \leqslant \tfrac{1}{6}C^3\omega_3(\Delta s)(s - s_0) \cdot 2(3s_1 - s_0 - 2s) \leqslant \tfrac{1}{3}C^3\omega_3(\Delta s)\,\Delta s^2;$$

by (1.11),

$$|L_4| \leqslant \tfrac{1}{2}C\omega_3(\Delta s)(s - s_0)(2s_1 - s_0 - s) \leqslant \tfrac{3}{8}C\omega_3(\Delta s)\,\Delta s^2;$$

$$|L_5| \leqslant \tfrac{1}{8}\omega_3(\Delta s)\,\Delta s^2.$$

Hence we have

$$\begin{aligned}|f'(t) - S'(t)| &\leqslant \frac{1}{8}\omega_3\left(\frac{\Delta s}{2}\right)\Delta s^2 + \left(\frac{1}{8} + \frac{3}{8}C + \frac{1}{3}C^3\right)\omega_3(\Delta s)\,\Delta s^2\\ &\leqslant \left(\frac{1}{4} + \frac{3}{8}C + \frac{1}{3}C^3\right)\omega_3(\Delta s)\,\Delta s^2.\end{aligned}$$

We then rewrite (1.4) as

$$\begin{aligned}f''(t) - S''(t) &= \frac{2}{\Delta t^3}\left(\Delta y - y_0'\Delta t - \frac{y_0''}{2}\Delta t^2 - \frac{y_0'''}{6}\Delta t^3\right)(3t - t_0 - 2t_1)\\ &\quad + \frac{2}{\Delta t^3}\left(\Delta y - y_1'\Delta t + \frac{y_1''}{2}\Delta t^2 - \frac{y_1'''}{6}\Delta t^3\right)(3t - 2t_0 - t_1)\\ &\quad - [f''(t) - y_0'' - y_0'''(t - t_0)]\,\frac{3t - t_0 - 2t_1}{\Delta t}\\ &\quad + [f''(t) - y_1'' - y_1'''(t - t_1)]\,\frac{3t - 2t_0 - t_1}{\Delta t}\\ &\quad + \frac{y_1''' - y_0'''}{3\Delta t}(3t - t_0 - 2t_1)(3t - 2t_0 - t_1)\\ &= M_1 + M_2 + M_3 + M_4 + M_5.\end{aligned}$$

Suppose $t \in \widehat{t_0 t}^*$; then, as in (1.12),

$$|M_1| + |M_2| \leqslant C^3\omega_3(\Delta s)\,\Delta s;$$

$$|M_3| \leqslant C\omega_3\left(\frac{\Delta s}{2}\right)\Delta s;$$

$$|M_4| \leqslant \tfrac{3}{2}C\omega_3(\Delta s)\,\Delta s;$$

$$|M_5| \leqslant \frac{1}{3}\omega_3(\Delta s)\frac{1}{|\Delta t|}(2s_1 - s_0 - s)(s_1 - 2s_0 + s) \leqslant \frac{3}{4}C\omega_3(\Delta s)\,\Delta s.$$

Hence we obtain

$$|f''(t)-S''(t)| \leqslant C\omega_3\left(\frac{\Delta s}{2}\right)\Delta s+\left(\frac{9}{4}C+C^3\right)\omega_3(\Delta s)\,\Delta s$$
$$\leqslant\left(\frac{13}{4}C+C^3\right)\omega_3(\Delta s)\,\Delta s.$$

2. Estimations in General Interpolation

Now let $L\colon t=t(s)$ be a smooth curve, closed or not. A function $f(t)\in C^1$ is given on L. We subdivide L by a mesh

$$\Delta\colon t_0=t(s_0),\qquad t_1=t(s_1),\dots,t_N=t(s_N)$$
$$(0=s_0<s_1<\cdots<s_N=L;\ t_N=t_0 \text{ in case } L \text{ is closed}).$$

The cubic spline with deficiency 2 interpolating $f(t)$ and $f'(t)$ at the knots $t_0,\dots,t_N$ is given by, as in (1.1),

$$S(t)\equiv S_j(t)=y_j+\frac{\Delta y_j}{\Delta t_j}(t-t_j)+\left(y_j'-\frac{\Delta y_j}{\Delta t_j}\right)\frac{(t-t_j)(t-t_{j+1})^2}{\Delta t_j^2}$$
$$+\left(y_{j+1}'-\frac{\Delta y_j}{\Delta t_j}\right)\frac{(t-t_j)^2\,(t-t_{j+1})}{\Delta t_j^2},\qquad t\in L_j=\widehat{t_jt}_{j+1},\quad (2.1)$$

where the notations are obvious. All the estimations in Section 1 are applicable on each L_j with C defined there. In order to get more accurate estimates, we may define

$$C_\delta=\sup_{|\widehat{tt'}|\leqslant\delta}|\widehat{tt'}|/|t-t'|\qquad (t,t'\in L)$$

($0<\delta\leqslant L$ when L is open, $0<\delta\leqslant L/2$ when L is closed). Obviously $C_\delta\downarrow 1$ when $\delta\downarrow 0$. Therefore, if we denote $\delta=\max_j\Delta s_j$ ($\Delta s_j=|\widehat{t_jt}_{j+1}|$) then the constant C which occurred in all estimations in section 1 may be replaced by C_δ.

Thus, owing to the results obtained in Section 1, we have the following theorems.

Theorem 1. *If $f(t)\in C^1$, the following estimates are valid:*

$$|f(t)-S(t)|\leqslant(\tfrac{1}{2}C_\delta+\tfrac{1}{4}C_\delta^3)\,\omega_1(\delta)\,\delta,$$
$$|f'(t)-S'(t)|\leqslant(C_\delta+\tfrac{3}{2}C_\delta^3)\,\omega_1(\delta).$$

THEOREM 2. *If $f(t) \in C^2$, the following estimates are valid*:

$$|f(t)-S(t)| \leqslant \frac{1}{8}\omega_2\left(\frac{\delta}{2}\right)\delta^2+\left(\frac{1}{16}C_\delta+\frac{1}{4}C_\delta^3\right)\omega_2(\delta)\,\delta^2$$

$$\leqslant\left(\frac{1}{8}+\frac{1}{16}C_\delta+\frac{1}{4}C_\delta^3\right)\omega_2(\delta)\,\delta^2,$$

$$|f'(t)-S'(t)| \leqslant \frac{1}{2}\omega_2\left(\frac{\delta}{2}\right)\delta+\left(\frac{3}{8}C_\delta+C_\delta^3\right)\omega_2(\delta)\,\delta$$

$$\leqslant\left(\frac{1}{2}+\frac{3}{8}C_\delta+C_\delta^3\right)\omega_2(\delta)\,\delta,$$

$$|f''(t)-S''(t)| \leqslant 3C_\delta^3\omega_2(\delta).$$

THEOREM 3. *If $f(t) \in C^3$, the following estimates are valid*:

$$|f(t)-S(t)| \leqslant \frac{1}{48}\omega_3\left(\frac{\delta}{2}\right)\delta^3+\left(\frac{1}{48}+\frac{1}{16}C_\delta+\frac{1}{12}C_\delta^3\right)\omega_3(\delta)\,\delta^3$$

$$\leqslant\left(\frac{1}{24}+\frac{1}{16}C_\delta+\frac{1}{12}C_\delta^3\right)\omega_3(\delta)\,\delta^3,$$

$$|f'(t)-S'(t)| \leqslant \frac{1}{8}\omega_3\left(\frac{\delta}{2}\right)\delta^2+\left(\frac{1}{8}+\frac{3}{8}C_\delta+\frac{1}{3}C_\delta^3\right)\omega_3(\delta)\,\delta^2$$

$$\leqslant\left(\frac{1}{4}+\frac{3}{8}C_\delta+\frac{1}{3}C_\delta^3\right)\omega_3(\delta)\,\delta^2,$$

$$|f''(t)-S''(t)| \leqslant C_\delta\omega_3\left(\frac{\delta}{2}\right)\delta+\left(\frac{9}{4}C_\delta+C_\delta^3\right)\omega_3(\delta)\,\delta$$

$$\leqslant\left(\frac{13}{4}C_\delta+C_\delta^3\right)\omega_3(\delta)\,\delta,$$

$$|f'''(t)-S'''(t) \leqslant (3C_\delta+2C_\delta^3)\,\omega_3(\delta).$$

THEOREM 4. *If $f(t) \in C^r$ $(r=1,2,3)$, then $S^{(p)}(t)$ tends uniformly to $f^{(p)}(t)$ $(p=0,\dots,r)$ when $\delta \to 0$.*

THEOREM 5. *If $f^{(r)}(t) \in H^\alpha$ (Hölder condition), then*

$$|f^{(p)}(t)-S^{(p)}(t)| = O(\delta^{\alpha+r-p}) \qquad (0 \leqslant p \leqslant r,\ r=1,2,3).$$

Remark. If $f^{(r)}(t)$ is arc-wise continuous, then these theorems remain true, provided the points of discontinuity are taken as a subset of the knots. If the curve L is arc-wise smooth, we may treat it separately.

3. Case of Continuous Functions

When $f(t)$ does not possess a continuous derivative and is simply continuous on L, we may proceed as follows.

Take a mesh (2.1) of L. First of all, we construct a function $F(t)$ linearly interpolating $f(t)$ at the knots t_j, i.e.,

$$F(t) \equiv F_j(t) = y_j + \frac{\Delta y_j}{\Delta t_j}(t - t_j), \qquad t \in L_j.$$

Then, near each t_j, we take two points t_j', t_j'', respectively, on L_{j-1} and L_j [1] such that, for simplicity,

$$|\widehat{t_j' t_j}| = |\widehat{t_j t_j''}| = \lambda \min(\Delta s_{j-1}, \Delta s_j), \qquad \lambda \leqslant \tfrac{1}{2}.$$

Now we interpolate $F(t)$ on each $L_j' = \widehat{t_j' t_j''}$ as in Section 1 and obtain $F_j^*(t)$. Define

$$\begin{aligned} F^*(t) &= F_j^*(t), \qquad \text{when} \quad t \in L_j', \\ &= F(t), \qquad \text{when} \quad t \notin \bigcup L_j'. \end{aligned}$$

Thus, $F^*(t)$ is also a cubic spline of deficiency 2. Let us estimate $|f(t) - F^*(t)|$.

If $t \in \widehat{t_j'' t_{j+1}'}$, then

$$\begin{aligned} |f(t) - F^*(t)| &= \left| f(t) - y_j - \frac{\Delta y_j}{\Delta t_j}(t - t_j) \right| \\ &\leqslant \omega_0\left(\frac{\delta}{2}\right) + \frac{1}{2} C_\delta \omega_0(\delta) \\ &\leqslant \left(1 + \frac{1}{2} C_\delta\right) \omega_0(\delta), \end{aligned} \tag{3.1}$$

provided $t \in \widehat{t_j'' t_j^*}$. By symmetry, it is valid for $t \in \widehat{t_j'' t_{j+1}'}$ also.

[1] For brevity, we consider L as a closed contour. However, the method used here remains effective for an open arc.

If $t \in L'_j$, then, by (2.1),

$$F^*(t) \equiv F_j^*(t) = F_{j-1}(t'_j) + \left(\frac{\Delta F}{\Delta t}\right)_j (t - t'_j)$$
$$+ \left[F'_{j-1}(t'_j) - \left(\frac{\Delta F}{\Delta t}\right)_j\right] \frac{(t-t'_j)(t-t''_j)^2}{\Delta t_j'^2}$$
$$+ \left[F'_j(t''_j) - \left(\frac{\Delta F}{\Delta t}\right)_j\right] \frac{(t-t'_j)^2\,(t-t''_j)}{\Delta t_j'^2},$$

where

$$\Delta t'_j = t''_j - t'_j, \qquad \left(\frac{\Delta F}{\Delta t}\right)_j = \frac{F_j(t''_j) - F_{j-1}(t'_j)}{\Delta t'_j},$$

while

$$F_{j-1}(t'_j) = y_j + \frac{\Delta y_{j-1}}{\Delta t_{j-1}} (t'_j - t_j), \qquad F_j(t''_j) = y_j + \frac{\Delta y_j}{\Delta t_j} (t''_j - t_j),$$
$$F'_{j-1}(t'_j) = \frac{\Delta y_{j-1}}{\Delta t_{j-1}}, \qquad F'_j(t''_j) = \frac{\Delta y_j}{\Delta t_j}.$$

Suppose $t \in \widehat{t'_j t_j}$; then

$$|f(t) - F_{j-1}(t'_j)| = \left| f(t) - y_j - \frac{\Delta y_{j-1}}{\Delta t_{j-1}} (t'_j - t_j)\right|$$
$$\leqslant \omega_0(\lambda\delta) + \lambda C_\delta\, \omega_0(\delta)$$
$$\leqslant (1 + \lambda C_\delta)\, \omega_0(\delta), \tag{3.2}$$
$$\left| \left(\frac{\Delta F}{\Delta t}\right)_j (t - t'_j)\right| = \left| \frac{\Delta y_j}{\Delta t_j} (t''_j - t_j) - \frac{\Delta y_{j-1}}{\Delta t_{j-1}} (t'_j - t_j)\right| \cdot \left| \frac{t - t'_j}{\Delta t'_j}\right|$$
$$\leqslant \lambda C_\delta C_{2\lambda\delta}\, \omega_0(\delta) \leqslant \lambda C_\delta^2\, \omega_0(\delta).$$

Since

$$\left| F'_{j-1}(t'_j) - \left(\frac{\Delta F}{\Delta t}\right)_j \right| = \left| \frac{\Delta y_{j-1}}{\Delta t_{j-1}} - \frac{\Delta y_j}{\Delta t_j}\right| \cdot \left| \frac{t''_j - t_j}{\Delta t'_j}\right|$$
$$\leqslant \omega_0(\delta) \left(\frac{1}{|\Delta t_{j-1}|} + \frac{1}{|\Delta t_j|}\right) \left| \frac{t''_j - t_j}{\Delta t'_j}\right|$$
$$\leqslant 2\lambda C_\delta\, \omega_0(\delta) \cdot \frac{1}{|\Delta t'_j|}$$

and a similar estimation for $|F_j'(t_j'') - (\Delta F/\Delta t)_j|$ while

$$|(t - t_j')(t - t_j'')^2 + (t - t_j')^2 (t - t_j'')| \leqslant (s - s_j')(s_j'' - s) \Delta s_j' \leqslant \tfrac{1}{4} \Delta s_j'^3,$$

where $\Delta s_j' = s_j'' - s_j'$; hence we get

$$\begin{aligned} |f(t) - F^*(t)| &\leqslant \omega_0(\lambda\delta) + (\lambda C_\delta + \lambda C_\delta^2 + \tfrac{1}{2}\lambda C_\delta^4)\, \omega_0(\delta) \\ &\leqslant (1 + \lambda C_\delta + \lambda C_\delta^2 + \tfrac{1}{2}\lambda C_\delta^4)\, \omega_0(\delta), \end{aligned} \tag{3.3}$$

which is valid for every $t \in L$ by symmetry.

On comparing (3.3) with (3.1), we finally obtain, for every $t \in L$,

$$|f(t) - F^*(t)| \leqslant \max\{1 + \tfrac{1}{2}C_\delta,\, 1 + \lambda C_\delta + \lambda C_\delta^2 + \tfrac{1}{2}\lambda C_\delta^4\}\, \omega_0(\delta). \tag{3.4}$$

Thus, we have

THEOREM 6. *If $f(t) \in C$, then estimate* (3.4) *is valid; if $f(t) \in H^\alpha$, then*

$$|f(t) - F^*(t)| = O(\delta^\alpha).$$

When $\lambda \geqslant \frac{1}{5}$, it is easily verified (3.3) is valid for every $t \in L$.

Note that, from (3.2) and (3.3), it is readily seen

$$|f(t_j) - F^*(t_j)| \leqslant A\lambda\omega_0(\delta) \qquad (A = \text{const}).$$

This means the spline $F^*(t)$ is approximately interpolating $f(t)$ at t_j if we take λ sufficiently small.

Remark. If the knots t_j are uniformly distributed on L, or L is a line-segment or a circle (circular arc), all the estimations both in Section 2 and in Section 3 may be simplified and improved.

4. THE ESTIMATION OF C_δ

To complete our discussion, we should determine, or, at least, estimate the value C_δ for a given smooth curve L. When L is a line-segment, obviously $C_\delta = 1$; when L is a circular arc,

$$C_\delta = \frac{\delta}{2 \sin \delta/2}.$$

In general, we suppose L has a bounded curvature κ, say,

$$|\kappa| \leqslant K.$$

It is easy to show

$$1 \leqslant C_\delta \leqslant (1 - K\delta)^{-1/2} \qquad (0 < \delta < 1/K). \tag{4.2}$$

In fact, let $t(s) = x(s) + iy(s)$ and $t = t(s)$, $t^* = t(s^*)$ be two arbitrary points on L with $|\Delta s| = |s^* - s| \leqslant \delta$. Then

$$|\Delta t|^2 = |t^* - t|^2 = [x'(\sigma_1)^2 + y'(\sigma_2)^2]\,\Delta s^2 = (\cos^2\theta_1 + \sin^2\theta_2)\,\Delta s^2,$$

where $s = \sigma_1, \sigma_2$ are two points on the arc $\widehat{tt}\,^*$ and θ_1, θ_2 are the inclinations of the tangents of L at σ_1, σ_2, respectively. Therefore

$$\left|\frac{\Delta t}{\Delta s}\right|^2 = 1 - \sin(\theta_1 + \theta_2)\sin(\theta_1 - \theta_2) \geqslant 1 - |\Delta\theta|,$$

where $|\Delta\theta| = |\theta_2 - \theta_1|$, and thereby

$$\left|\frac{\Delta t}{\Delta s}\right|^2 \geqslant 1 - \left|\frac{\Delta\theta}{\Delta\sigma}\right| |\Delta\sigma| \geqslant 1 - K|\Delta\sigma| \qquad (|\Delta\sigma| = |\sigma_1 - \sigma_2|),$$

from which (4.2) follows immediately.

Estimate (4.2) is useful, especially when δ is small. However, it is very rough. For example, if L is the unit circle, then, from (4.2), we have $C_\delta \leqslant (1-\delta)^{-1/2}$ which is much greater than the exact value given by (4.1).

We also point out that, in many cases, C_δ decreases to 1 very rapidly when $\delta \to 0$; e.g., on the unit circle, $C_\delta \approx 1.016$ when $\delta = \pi/5$, so that, for sufficiently small δ, we may take $C_\delta \approx 1$ for simplification during practical calculations.

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